1 Vectors

Definition 1.1. *A directed quantity, as a straight line, a force, or a velocity. Vectors are said to be equal when their directions are the same, their magnitudes equal.*

Definition 1.2. *The length or magnitude or norm of the vector a is denoted by* $|\vec{a}|$.

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2}$$

Definition 1.3. *A unit vector is any vector with a length of one.*

1.1 Vector equality

Definition 1.4. *Two vectors are said to be equal if they have the same magnitude and direction. (However if we are talking about bound vector, then two bound vectors are equal if they have the same base point and end point.)*

For example, the vector $\vec{i} + 2\vec{j}$ with base point (1,0) and the vector $\vec{i} + 2\vec{j}$ with base point (0,1) are different bound vectors, but the same (unbounded) vector.

1.2 Vector addition and subtraction

Let $\vec{a} = a_1\vec{i} + a_2\vec{j}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j}$. The sum of $\vec{a}$ and $\vec{b}$ is:

Definition 1.5.

$$\vec{a} + \vec{b} = (a_1 + b_1) \cdot \vec{i} + (a_2 + b_2) \cdot \vec{j}$$

The addition may be represented graphically by placing the start of the arrow $\vec{b}$ at the tip of the arrow $\vec{a}$, and then drawing an arrow from the start of $\vec{a}$ to the tip of $\vec{b}$. The new arrow drawn represents the vector $\vec{a} + \vec{b}$.

This addition method is sometimes called the parallelogram rule because $\vec{a}$ and $\vec{b}$ form the sides of a parallelogram and $\vec{a} + \vec{b}$ is one of the diagonals. If $\vec{a}$ and $\vec{b}$ are bound vectors, then the addition is only defined if $\vec{a}$ and $\vec{b}$ have the same base point, which will then also be the base point of $\vec{a} + \vec{b}$. One can check geometrically that

Theorem 1.6.

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

and

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

.

The difference of a and b is:

Definition 1.7.

$$\vec{a} - \vec{b} = (a_1 - a_2)\vec{i} + (b_1 - b_2)\vec{j}$$

Subtraction of two vectors can be geometrically defined as follows: to subtract $\vec{b}$ from $\vec{a}$, place the ends of $\vec{a}$ and $\vec{b}$ at the same point, and then draw an arrow from the tip of $\vec{b}$ to the tip of $\vec{a}$. That arrow represents the vector $\vec{a} - \vec{b}$.

Definition 1.8.

$$\overrightarrow{AB} = \vec{b} - \vec{a}$$

If $\vec{a}$ and $\vec{b}$ are bound vectors, then the subtraction is only defined if they share the same base point which will then also become the base point of their difference. This operation deserves the name "subtraction" because $(\vec{a}\vec{b}) + \vec{b} = \vec{a}$.

1.3 Scalar multiplication

A vector may also be multiplied by a real number r. In mathematics numbers are often called scalars to distinguish them from vectors, and this operation is therefore called scalar multiplication. The resulting vector is:

$$r\vec{a} = (ra_1)\vec{i} + (ra_2)\vec{j}$$

The length of $r \cdot \vec{a}$ is $|r| \cdot |\vec{a}|$. If the scalar is negative, it also changes the direction of the vector by 180°.

Here it is important to check that the scalar multiplication is compatible with vector addition in the following sense: $r(\vec{a} + \vec{b}) = r\vec{a} + r\vec{b}$ for all vectors $\vec{a}$ and $\vec{b}$ and all scalars r. One can also show that $\vec{a} - \vec{b} = \vec{a} + (-1)\vec{b}$.

1.4 Dot product

Definition 1.9. *The dot product of two vectors $\vec{a}$ and $\vec{b}$ (sometimes called inner product, or, since its result is a scalar, the scalar product) is denoted by $\vec{a} \cdot \vec{b}$ and is defined as:*

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \gamma$$

where $|\vec{a}|$ and $|\vec{b}|$ denote the norm (or length) of $\vec{a}$ and $\vec{a}$, and γ is the measure of the angle between $\vec{a}$ and $\vec{a}$ (see trigonometric function for an explanation of cosine). Geometrically, this means that $\vec{a}$ and $\vec{b}$ are drawn with a common start point and then the length of $\vec{a}$ is multiplied with the length of that component of $\vec{a}$ that points in the same direction as $\vec{a}$.

The dot product can also be defined as the sum of the products of the components of each vector:

$$\vec{a} \cdot \vec{b} = a_1 \cdot b_1 + a_2 \cdot b_2$$

where $\vec{a}$ and $\vec{b}$ are vectors a_1, a_2 and b_1, b_2 are coordinates of $\vec{a}$ and $\vec{b}$.

Theorem 1.10. *The dot product equals 0 if and only if the angle between the two vectors is* $90°$.